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The essential rank of classical groups

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ABSTRACT

Let G be a finite classical group defined over a finite field with odd characteristic. Let $r > 2$ be a prime, not dividing the characteristic, and $D \leqslant G$ a Sylow r-subgroup. We consider the Frobenius category $\mathcal{F}_D(G)$ and determine the cardinality of a minimal conjugation family for it. This amounts to determining the number of G-conjugacy classes of essential subgroups of D, that is, the essential rank of $\mathcal{F}_D(G)$. In addition, we characterise those classical groups which allow an r-local subgroup controlling r-fusion.

1. Introduction

In 1967 Alperin [1] introduced the notion of a conjugation family and proved that the fusion of non-trivial subgroups of a Sylow subgroup is completely determined by their normalisers. Alperin's Fusion Theorem has been refined by Goldschmidt [9], which is known as the Alperin–Goldschmidt conjugation family. Many modifications of this conjugation family exist and we refer to the book of Huppert and Blackburn [8, Section X.4] for some more information. The definition of a conjugation family for a Sylow subgroup used in this paper is as follows.

Definition 1.1. Let G be a finite group with Sylow r-subgroup D. A *conjugation family* for D is a set $\mathcal{C}$ of subgroups of D such that for all non-trivial $A \leqslant D$ and $g \in G$ with $A^g \leqslant D$ there exist $Q_1, \dots, Q_n \in \mathcal{C}$ and elements $g_i \in N_G(Q_i)$ and $c \in C_G(A)$ with $g = cg_1 \dots g_n$, $A \leqslant Q_1$, and $A^{g_1 \cdots g_{i-1}} \leqslant Q_i$ for all $i \in \{2, \dots, n\}$.

As an abstract model for the r-local structure of a group, Puig [12] gave an axiomatic description of fusion systems. Conjugation families are also defined for fusion systems, and the minimal possible

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cardinality of such a family is by one larger than the essential rank of the system. An example for a fusion system is the Frobenius category on an r-subgroup.

This paper is part of a research project investigating the essential rank of finite groups. In [3] the essential rank of the Frobenius category on r-subgroups of symmetric groups is determined. Sporadic simple groups are considered in [4], and Puig [12, Appendix I] considered finite groups of Lie type in defining characteristic. The aim of this paper is to determine a formula for the essential rank of the Frobenius category $\mathcal{F}_D(G)$ where G is a classical group defined over a finite field with odd characteristic p and $D \leqslant G$ is a Sylow r-subgroup with $r \notin \{2, p\}$. In addition, we characterise those classical groups which allow an r-local subgroup controlling r-fusion; for symmetric groups this is considered in [3]. For the sake of completeness, we recall all definitions.

1.1. Frobenius categories

Let P be an r-subgroup of a finite group G. We first describe the definition of a fusion system on P as given in [10, Section 2]. For subgroups $U, V, W \leqslant G$ let $\mathrm{Hom}_W(U, V)$ be the set of group homomorphisms $\varphi : U \to V$ for which there exists $w \in W$ such that $\varphi(u) = w^{-1}uw$ for all $u \in U$. Consequently, we write $\mathrm{Aut}_W(U) = \mathrm{Hom}_W(U, U)$.

A *category* $\mathcal{F}$ on P has all subgroups of P as objects, and for $Q, R \leqslant P$ the morphism set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ consists of injective group homomorphisms $Q \to R$ with the following properties. First, if $Q \leqslant R$, then the inclusion $Q \hookrightarrow R$ is a morphism. Second, if $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then the induced isomorphism $Q \cong \varphi(Q)$ and its inverse are morphisms. The composition of morphisms is the usual composition of group homomorphisms.

A *fusion system* on P is a category $\mathcal{F}$ on P such that $\mathrm{Hom}_P(Q, R)$ is contained in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ for all $Q, R \leqslant P$, and such that the following hold. First, $\mathrm{Aut}_P(P)$ is a Sylow r-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Hom}_{\mathcal{F}}(P, P)$. Second, every $\mathcal{F}$-isomorphism $\varphi : Q \to R$ such that $|N_P(R)|$ is maximal in the $\mathcal{F}$-isomorphism class of R extends to an element of $\mathrm{Hom}_{\mathcal{F}}(N_P(Q), N_P(R))$ up to modifying φ by an element of $\mathrm{Aut}_{\mathcal{F}}(R)$; cf. [10, Lemma 2.6].

The *Frobenius category* $\mathcal{F}_P(G)$ on P is the category on P whose morphisms are $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q, R) = \mathrm{Hom}_G(Q, R)$ for $Q, R \leqslant P$. That $\mathcal{F}_P(G)$ is a fusion system is shown in [10, Theorem 2.11].

1.2. Conjugation families

Let G be a finite group with r-subgroup P, and $\mathcal{F}$ a fusion system on P. We define a *conjugation family* for $\mathcal{F}$ as in [7, Definition 2.12]; observe that this coincides with Definition 1.1 if $\mathcal{F}$ is the Frobenius category on a Sylow subgroup.

Definition 1.2. Let G be a finite group with r-subgroup P. A *conjugation family* for a fusion system $\mathcal{F}$ on P is a set $\mathcal{C}$ of subgroups of P with the following property. For every $\mathcal{F}$-isomorphism $\varphi : Q \to R$ there exist subgroups $Q = A_0, A_1, \ldots, A_n = R \leqslant P$ and elements $Q_1, \ldots, Q_n \in \mathcal{C}$ such $A_{i-1}, A_i \leqslant Q_i$ for all $i \geqslant 1$, and there exist $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$ such that $\alpha_i(A_{i-1}) = A_i$ for all $i \geqslant 1$, and $\varphi = \alpha_n|_{A_{n-1},A_n} \circ \cdots \circ \alpha_2|_{A_1,A_2} \circ \alpha_1|_{A_0,A_1}$.

To state a necessary and sufficient condition on $\mathcal{C}$ to be a conjugation family for $\mathcal{F}$, we need more notation. A proper subgroup U of G is strongly r-embedded if $r \mid |U|$ and $r \nmid |U \cap U^g|$ for all $g \in G \setminus U$. A subgroup Q of P is $\mathcal{F}$-essential if $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly r-embedded subgroup and Q is $\mathcal{F}$-centric, that is, whenever $\varphi : Q \to R$ is an isomorphism in $\mathcal{F}$, then $C_P(R) = Z(R)$, the center of R. Two subgroups $Q, R \leqslant P$ are $\mathcal{F}$-conjugate if there is an isomorphism $\varphi : Q \to R$ in $\mathcal{F}$. The class of essential groups is closed under $\mathcal{F}$-conjugacy, and an $\mathcal{F}$-conjugacy class of an essential subgroup is called an essential class of $\mathcal{F}$. Alperin's Fusion Theorem [7, Theorem 2.13] and the comments on [7, p. 686] imply the following.

Theorem 1.3. *Let $\mathcal{C}$ be a set of subgroups of P. Then $\mathcal{C}$ is a conjugation family for $\mathcal{F}$ if and only if $P \in \mathcal{C}$ and $\mathcal{C}$ intersects non-trivially with every essential class.*

Hence, the minimal cardinality of a conjugation family for $\mathcal{F}$ is by one larger than the number of $\mathcal{F}$-conjugacy classes of essential subgroups. As in [7] (and [3,4]), we call the number of essential classes the essential rank $\mathrm{rk}_e(\mathcal{F})$ of $\mathcal{F}$. If $\mathcal{F}$ is the Frobenius category on P, then $Q \leqslant P$ is $\mathcal{F}$-essential if and only if $Z(Q)$ is a Sylow r-subgroup of $C_G(Q)$, and $N_G(Q)/QC_G(Q)$ has a strongly r-embedded subgroup. In this case, one also says that Q is an essential subgroup of G.

1.3. Main results

We need some number theoretic definitions to describe our first main result. Let r be a prime. For an integer $m > 0$ denote by $m = \sum_{i=0}^{\ell} m_i r^i$ its r-adic expansion with $m_\ell \neq 0$. The reduced Hamming weight of m and its complement are

$$H(m) = \bigl|\bigl\{i \in \{1, \dots, \ell\} \bigm| m_i \neq 0\bigr\}\bigr| \quad \text{and} \quad \overline{H(m)} = \bigl|\bigl\{i \in \{1, \dots, \ell\} \bigm| m_i = 0\bigr\}\bigr|.$$

If $\ell \geqslant 2$, $m_{\ell-1} = 0$, and $m_\ell = 1$, then let $Y(m) = -1$, and $Y(m) = 0$ otherwise. For an integer $n \neq 0$ let n_r be the largest r-power dividing n. The next definition is as in [2, p. 13] and [5, p. 6], and defines certain parameters for a classical group G and prime r.

Definition 1.4. Let q be an odd prime power, and $r > 2$ a prime not dividing q. Write $G = \mathrm{G}(d, q)$ for any of $\mathrm{GL}(d, q)$, $\mathrm{GU}(d, q)$, $\mathrm{Sp}(2d, q)$, $\mathrm{O}^0(2d + 1, q)$ and $\mathrm{O}^{\pm}(2d, q)$, and call $\mathrm{G}(d, q)$ of *linear*, *unitary*, *symplectic*, *orthogonal* 0, and *orthogonal* $\pm$ *type*, respectively. If G is linear or unitary, then let e be the order of q and $-q$ modulo r, respectively. For all other types let e be the order of q^2 modulo r. If G is linear, then let $r^a = (q^e - 1)_r$, and $r^a = (q^{2e} - 1)_r$ for all other types. The sign $\varepsilon = \pm 1$ is defined such that $r \mid q^e - \varepsilon$. Let D be a Sylow r-subgroup of G with 1-eigenspace of dimension f. If G is linear or unitary, then let $d_0 = f$. If G is symplectic or orthogonal with even degree, then $d_0 = f/2$. For orthogonal 0 type define $d_0 = (f - 1)/2$. We say $d, d_0, q, r, e, a, \varepsilon$ are *parameters* of G.

Remark 2.5 shows that d_0 is always an integer. The next result is proved in Section 3.

Theorem 1.5. *Let $G = \mathrm{G}(d, q)$ with parameters $d, d_0, q, r, e, a, \varepsilon$ and Sylow r-subgroup D. Write $d = d_0 + ew$ where w has r-adic decomposition $w = w_0 + w_1 r + \cdots + w_\ell r^\ell$ with $w_\ell \neq 0$. If $\ell = 0$, then $D = \{1\}$ and $\mathrm{rk}_e(\mathcal{F}_D(G)) = 0$, otherwise $\mathrm{rk}_e(\mathcal{F}_D(G)) = \ell(\ell + 3)/2 + \overline{H(m)} + Y(m) - 1$.*

A subgroup $K \leqslant G$ is r-local if $K = N_G(P)$ for some non-trivial r-subgroup $P \leqslant G$. Let $G = \mathrm{G}(d, q)$ with Sylow r-subgroup D. An r-local subgroup $K \leqslant G$ controls r-fusion if $D \leqslant K$ and for all $Q, Q^g \leqslant D$ with $g \in G$ there exist $c \in C_G(Q)$ and $k \in K$ with $g = ck$. We prove the following in Section 4.

Theorem 1.6. *Let $G = \mathrm{G}(d, q)$ with parameters $d, d_0, q, r, e, a, \varepsilon$ and Sylow r-subgroup D. There exists an r-local subgroup $K \leqslant G$ which controls r-fusion if and only if D is abelian, or $e = 1$ and G is linear or unitary.*

2. Radical subgroups of classical groups

A subgroup $R \leqslant G$ is r-radical if $R = O_r(N_G(R))$, the largest normal r-subgroup of $N_G(R)$. Clearly, every essential subgroup is radical. Alperin and Fong [2] determined the structure of the radical subgroups of symmetric and general linear groups. An [5] considered the radical subgroups of the unitary, symplectic, and orthogonal groups. This section recalls some of their results; we refer to [2] and [5] for details and proofs.

First, we need some notation on symmetric groups. For an integer n let $S(n)$ be the symmetric group on $\{1, \dots, n\}$. Let $X \leqslant S(i)$ be fixed point free with underlying set $W = \{1, \dots, i\}$ and let $Y \leqslant S(j)$ be transitive. The permutational wreath product $Z = X \wr Y \leqslant S(ij)$ is $Z = (X_1 \times \cdots \times X_j) \rtimes Y$, where each $X_k = X$ has underlying set $W_k = W$ and Y permutes $\{X_1, \dots, X_j\}$ according to its natural action on $\{1, \dots, j\}$.

Definition 2.1. Let r be a prime and let $c_1, \dots, c_t, d$ be positive integers.

(a) Let A_d be the elementary abelian group of order r^d, considered as a transitive subgroup of $S(r^d)$ via its regular permutation action.
(b) For $\mathbf{c} = (c_1, \dots, c_t)$ define $A_{\mathbf{c}} = A_{c_1} \wr A_{c_2} \wr \cdots \wr A_{c_t} \leqslant S(r^{|\mathbf{c}|})$ where $|\mathbf{c}| = c_1 + \cdots + c_t$ is the weight of $\mathbf{c}$. The group $A_{\mathbf{c}}$ is a *basic subgroup* of $S(r^{|\mathbf{c}|})$.

2.1. General linear and unitary groups

Let $G = \mathrm{G}(d,q)$ be linear or unitary with parameters $d, d_0, q, r, e, a, \varepsilon$, see Definition 1.4, and natural module V. We also write $\mathrm{GL}^+(d,q)$ and $\mathrm{GL}^-(d,q)$ for $\mathrm{GL}(d,q)$ and $\mathrm{GU}(d,q)$, respectively. For a linear (unitary) space U we denote by $\mathrm{GL}^{\pm}(U)$ the group of linear (unitary) transformations of U.

Definition 2.2. For $\alpha \geqslant 0$ let Z_α be a cyclic group of order $r^{a+\alpha}$. For $\gamma \geqslant 0$ let E_γ be an extraspecial group of order $r^{1+2\gamma}$. The central product $Z_\alpha E_\gamma$ over $\Omega_1(Z_\alpha) = Z(E_\gamma)$ can be embedded into $\mathrm{GL}^{\varepsilon}(r^\gamma, q^{er^\alpha})$. If G is linear, then we write $R_{\alpha,\gamma}$ for the image of $Z_\alpha E_\gamma$ under the Galois embedding $\mathrm{GL}(r^\gamma, q^{er^\alpha}) \hookrightarrow \mathrm{GL}(er^{\alpha+\gamma}, q)$. For $m \geqslant 1$ let $R_{m,\alpha,\gamma}$ be the m-fold diagonal embedding of $R_{\alpha,\gamma}$ in $\mathrm{GL}(mr^{\alpha+\gamma}, q)$. For a list of positive integers $\mathbf{c} = (c_1, \dots, c_t)$ define

$$R_{m,\alpha,\gamma,\mathbf{c}} = R_{m,\alpha,\gamma} \wr A_{\mathbf{c}},$$

considered as a subgroup of $\mathrm{GL}(d,q)$ where $d = mer^{\alpha+\gamma+c_1+\cdots+c_t}$. If G is unitary, then $R_{\alpha,\gamma}$, $R_{m,\alpha,\gamma}$, and $R_{m,\alpha,\gamma,\mathbf{c}} \leqslant \mathrm{GL}^-(d,q)$ are defined analogously. In both cases, $R_{m,\alpha,\gamma,\mathbf{c}}$ is determined up to conjugacy; we call it a *basic subgroup* of $\mathrm{GL}^{\pm}(d,q)$.

The next theorem is [5, (2B)] and [2, (4A)]. We write $U \perp W$ for the orthogonal sum of two unitary, symplectic, or orthogonal spaces. For linear spaces define $U \perp W = U \oplus W$.

Theorem 2.3. *If R is a radical subgroup of $G = \mathrm{GL}^{\pm}(d,q)$, then there are decompositions*

$$V = V_0 \perp \cdots \perp V_s \quad \textit{and} \quad R = R_0 \times \cdots \times R_s$$

such that R_0 is the trivial subgroup of $\mathrm{GL}^{\pm}(V_0)$, and R_i is a basic subgroup of $\mathrm{GL}^{\pm}(V_i)$ whose extraspecial components have exponent r for $i \geqslant 1$.

An r-subgroup R of G is of radical type if it admits decompositions as in Theorem 2.3; note that a group of radical type is not necessarily radical.

We use the notation of Theorem 2.3 and define

$$V(m,\alpha,\gamma,\mathbf{c}) = \sum_i V_i \quad \text{and} \quad R(m,\alpha,\gamma,\mathbf{c}) = \prod_i R_i,$$

where i runs over the indices with $R_i = R_{m,\alpha,\gamma,\mathbf{c}}$. If $G(m,\alpha,\gamma,\mathbf{c}) = \mathrm{GL}^{\pm}(V(m,\alpha,\gamma,\mathbf{c}))$, then

$$N_G(R) = \mathrm{GL}^{\pm}(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{G(m,\alpha,\gamma,\mathbf{c})}\big(R(m,\alpha,\gamma,\mathbf{c})\big),$$

$$N_G(R)/R = \mathrm{GL}^{\pm}(V_0) \times \prod_{m,\alpha,\gamma,\mathbf{c}} N_{G(m,\alpha,\gamma,\mathbf{c})}\big(R(m,\alpha,\gamma,\mathbf{c})\big)/R(m,\alpha,\gamma,\mathbf{c}).$$

Let $S = R(m,\alpha,\gamma,\mathbf{c})$ and $U = V(m,\alpha,\gamma,\mathbf{c})$ in the following. If S is a direct product of t copies of $R_{m,\alpha,\gamma,\mathbf{c}}$, then

$$N_{\mathrm{GL}^{\pm}(U)}(S) = N_{\mathrm{GL}^{\pm}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}}) \wr S(t),$$
$$N_{\mathrm{GL}^{\pm}(U)}(S)/S = N_{\mathrm{GL}^{\pm}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}} \wr S(t),$$

where $V_{m,\alpha,\gamma,\mathbf{c}}$ is the underlying space of $R_{m,\alpha,\gamma,\mathbf{c}}$. If $N_{m,\alpha,\gamma}$ and $C_{m,\alpha,\gamma}$ denote the normaliser and centraliser of $R_{m,\alpha,\gamma}$ in $\mathrm{GL}^{\pm}(mer^{\alpha+\gamma}, q)$, respectively, then

$$N_{\mathrm{GL}^{\pm}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})/R_{m,\alpha,\gamma,\mathbf{c}} \cong N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma} \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_t, r),$$
$$C_{m,\alpha,\gamma} \cong \mathrm{GL}^{\varepsilon}\bigl(m, q^{er^{\alpha}}\bigr), \quad \text{and}$$
$$C_{\mathrm{GL}^{\pm}(U)}(S) \cong \bigl(C_{\mathrm{GL}^{\pm}(V_{m,\alpha,\gamma,\mathbf{c}})}(R_{m,\alpha,\gamma,\mathbf{c}})\bigr)^{t} \cong (C_{m,\alpha,\gamma})^{t}.$$

If $N^0_{m,\alpha,\gamma} = \{g \in N_{m,\alpha,\gamma} \mid [g, Z(R_{m,\alpha,\gamma})] = 1\}$, then $N_{m,\alpha,\gamma}/N^0_{m,\alpha,\gamma} \cong C_{er^{\alpha}}$, the cyclic group of order er^{α}. If G is linear, then

$$N^0_{m,\alpha,\gamma} = L_{m,\alpha,\gamma} C_{m,\alpha,\gamma} R_{m,\alpha,\gamma}$$

where $L_{m,\alpha,\gamma} \cong \mathrm{Sp}(2\gamma, r)$ with $L_{m,\alpha,\gamma} \cap C_{m,\alpha,\gamma} R_{m,\alpha,\gamma} = 1$, commuting with $C_{m,\alpha,\gamma}$. If G is unitary, then $N^0_{m,\alpha,\gamma} = L_{m,\alpha,\gamma} C_{m,\alpha,\gamma}$ where $L_{m,\alpha,\gamma}$ contains $R_{m,\alpha,\gamma}$, commutes with $C_{m,\alpha,\gamma}$, and satisfies $L_{m,\alpha,\gamma} \cap C_{m,\alpha,\gamma} = Z(C_{m,\alpha,\gamma}) = Z(L_{m,\alpha,\gamma})$ and $L_{m,\alpha,\gamma}/Z(L_{m,\alpha,\gamma})R_{m,\alpha,\gamma} \cong \mathrm{Sp}(2\gamma, r)$. In both cases,

$$X_{m,\alpha,\gamma} = N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma} C_{m,\alpha,\gamma}$$

is isomorphic to an extension of $\mathrm{Sp}(2\gamma, r)$ by $C_{er^{\alpha}}$. If $\mathbf{c} = (c_1, \ldots, c_k)$, then

$$N_{\mathrm{GL}^{\pm}(U)}(S)/C_{\mathrm{GL}^{\pm}(U)}(S)S \cong \bigl[X_{m,\alpha,\gamma} \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_k, r)\bigr] \wr S(t). \tag{2.1}$$

Remark 2.4. An r-subgroup R of $G = \mathrm{GL}^{\pm}(d, q)$ of radical type can be decomposed as

$$R = R_0 \times R_1 \times \cdots \times R_s \quad \text{where } R_i = (R_{m_i,\alpha_i,\gamma_i,\mathbf{c}_i})^{t_i}$$

for pairwise distinct parameters $(m_i, \alpha_i, \gamma_i, \mathbf{c}_i)$. Let $V = V_0 \perp \cdots \perp V_s$ be the corresponding decomposition of V. The previous results show that

$$N_G(R)/RC_G(R) \cong \mathrm{GL}^{\pm}(V_0) \times H_1 \times \cdots \times H_s \tag{2.2}$$

where each $H_i = H_{m_i,\alpha_i,\gamma_i,\mathbf{c}_i}$ is of the form (2.1) with $t = t_i$, and

$$C_G(R) \cong \mathrm{GL}^{\pm}(V_0) \times (C_{m_1,\alpha_1,\gamma_1})^{t_1} \times \cdots \times (C_{m_s,\alpha_s,\gamma_s})^{t_s}.$$

2.2. Symplectic and orthogonal groups

In this section let $G = \mathrm{G}(d, q)$ be symplectic or orthogonal. For a symplectic (orthogonal) space U denote by $\mathrm{I}(U)$ the group of symplectic (orthogonal) isometries. Let $Z_\alpha E_\gamma$ be defined as in Definition 2.2. It is shown in [5, p. 12] that $Z_\alpha E_\gamma$ can be embedded in a group of symplectic (orthogonal) isometries of degree $2er^{\alpha+\gamma}$ defined over the field with q elements. As in the linear and unitary case, the basic subgroup $R_{m,\alpha,\gamma,\mathbf{c}} \leqslant \mathrm{I}(U)$ is defined where U is a symplectic (orthogonal) space of dimension $2mer^{\alpha+\gamma+|\mathbf{c}|}$; again, $R_{m,\alpha,\gamma,\mathbf{c}}$ is unique up to conjugacy. It is shown in [5, (2D)] that Theorem 2.3 also holds for G where $\mathrm{GL}^{\pm}(V_0)$ is replaced by $\mathrm{I}(V_0)$. Similarly, every subgroup of G of radical type can be decomposed as in Remark 2.4, where $\mathrm{GL}^{\pm}(V_0)$ is replaced by $\mathrm{I}(V_0)$, and in (2.1) the group $X_{m,\alpha,\gamma}$ is an extension of $\mathrm{Sp}(2\gamma, r)$ by C_{2er^α}, see [5, pp. 12 and 13]. The groups $N_{m,\alpha,\gamma}$ and $N^0_{m,\alpha,\gamma}$ are defined analogously to the unitary case with the difference that now $N_{m,\alpha,\gamma}/N^0_{m,\alpha,\gamma}$ is cyclic of order $2er^\alpha$ (and not er^α).

Remark 2.5. An r-subgroup R of $G = \mathrm{G}(d, q)$ of radical type can be decomposed as

$$R = R_0 \times R_1 \times \cdots \times R_s \quad \text{where } R_i = (R_{m_i,\alpha_i,\gamma_i,\mathbf{c}_i})^{t_i}$$

for pairwise distinct parameters $(m_i, \alpha_i, \gamma_i, \mathbf{c}_i)$. Let $V = V_0 \perp \cdots \perp V_s$ be the corresponding decomposition of V. As in Remark 2.4 we have

$$N_G(R)/RC_G(R) \cong \mathrm{I}(V_0) \times H_1 \times \cdots \times H_s \tag{2.3}$$

where each $H_i = H_{m_i,\alpha_i,\gamma_i,\mathbf{c}_i}$ is of the form (2.1) with $t = t_i$ and $X_{m,\alpha,\gamma}$ an extension of $\mathrm{Sp}(2\gamma, r)$ by C_{2er^α}. The centraliser is $C_G(R) \cong \mathrm{I}(V_0) \times (C_{m_1,\alpha_1,\gamma_1})^{t_1} \times \cdots \times (C_{m_s,\alpha_s,\gamma_s})^{t_s}$. By construction, every basic subgroup R_i acts on a space V_i of even dimension. Thus, if $\mathrm{G}(d, q) = O(2d+1, q)$, then the fixed point space V_0 of R has odd dimension.

3. Essential subgroups of classical groups

Throughout this section let $G = \mathrm{G}(d, q)$ with parameters $d, d_0, q, r, e, a, \varepsilon$, see Definition 1.4. Depending on the type of G, we denote by $\mathrm{G}(U)$ the group of all linear or unitary transformations, or symplectic or orthogonal isometries on the space U. Let D be a Sylow r-subgroup of G.

Lemma 3.1. *Let $R \leqslant G$ with $R \leqslant D$ be a radical subgroup as in Remark* 2.4 *or* 2.5. *Then R is essential if and only if R and D have the same* 1*-eigenspace, $m_1 = \cdots = m_s = 1$, and there exists $j \in \{1, \ldots, s\}$ such that one of the following holds*:

$$\begin{aligned}
&(1) \quad \alpha_j = 0, \quad \gamma_j = 0, \quad t_j = 1, \quad \mathbf{c}_j = (1, \ldots, 1, 2, 1, \ldots, 1),\\
&(2) \quad \alpha_j = 0, \quad \gamma_j = 0, \quad t_j \in \{r, \ldots, 2r\} \quad \mathbf{c}_j = (1, \ldots, 1),\\
&(3) \quad \alpha_j = 0, \quad \gamma_j = 1, \quad t_j = 1, \quad \mathbf{c}_j = (1, \ldots, 1),
\end{aligned}$$

and $\alpha_i = 0$, $\gamma_i = 0$, $t_i \in \{1, \ldots, r-1\}$, $\mathbf{c}_i = (1, \ldots, 1)$ for all $i \neq j$.
In particular, if $i \neq j$, then R_i is a Sylow r-subgroup of $\mathrm{G}(V_i)$.

Proof. If $Z(R)$ is a Sylow r-subgroup of $C_G(R)$, then $Z(D) \leqslant C_G(D) \leqslant C_G(R)$ shows that $Z(D) \leqslant Z(R)$. Remarks 2.4 and 2.5 imply that D and R have the same 1-eigenspace. Moreover, $Z(R_{m,\alpha,\gamma}) \cong Z_\alpha$ is a (cyclic) Sylow r-subgroup of $C_{m,\alpha,\gamma} \cong \mathrm{GL}^\varepsilon(m, q^{er^\alpha})$ if and only if $m = 1$; recall that r divides $q^{er^\alpha} - \varepsilon$. Thus we have proved that $Z(R)$ is a Sylow r-subgroup of $C_G(R)$ if and only $m_1 = \cdots = m_s = 1$, and R and D have the same 1-eigenspace. Assume R satisfies these conditions in the following.

It is easy to see that $N_G(R)/RC_G(R)$ has a strongly r-embedded subgroup if and only if exactly one direct factor in the decomposition (2.2) or (2.3) has a strongly r-embedded subgroup and all other factors are r'-groups, see for example [3, Lemma 10]. Clearly, a group of type (2.1) is of r'-order if and only if $\gamma = 0$, $\alpha = 0$, $\mathbf{c} = (1, \ldots, 1)$, and $t < r$. We now investigate when $H = H_i$ has a strongly r-embedded subgroup; for simplicity, assume that H is of the form (2.1). If $X_{m,\alpha,\gamma}$ is an r'-group, then the proof of [3, Lemma 11] shows that H has a strongly r-embedded subgroup if and only if $t = 1$ and $\mathbf{c} = (1, \ldots, 1, 2, 1, \ldots, 1)$, or $t \in \{r, \ldots, 2r\}$ and $\mathbf{c} = (1, \ldots, 1)$. Now suppose $X_{m,\alpha,\gamma}$ is not an r'-group. For a contradiction, suppose $t > 1$ and write the wreath product (2.1) as

$$X \wr S(t) = (X_1 \times \cdots \times X_t) \rtimes S(t)$$

where each X_l is a copy of $X = X_{m,\alpha,\gamma} \times \mathrm{GL}(c_1, r) \times \cdots \times \mathrm{GL}(c_k, r)$. Let $X_{l,r}$ be a Sylow r-subgroup of X_l, and let $S \leqslant X \wr S(t)$ be a Sylow r-subgroup containing every $X_{l,r}$. Clearly, $S(t) \leqslant N_H(X_{1,r} \times \cdots \times X_{t,r})$. Moreover, $X_1 \leqslant N_H(1 \times X_{2,r} \times \cdots \times X_{t,r})$ and similarly for $X_2, \ldots, X_t$. It follows that $H = \langle N_H(K) \mid 1 \neq K \leqslant S\rangle$. This is a contradiction to the assumption that H has a strongly r-embedded subgroup, see for example [3, Lemma 10b)], and $t = 1$ follows. Now H has a strongly r-embedded subgroup if and only if $\mathbf{c} = (1, \ldots, 1)$ and $X_{m,\alpha,\gamma}$ has a strongly r-embedded subgroup. By Aschbacher [6, (6.2)], the latter holds if and only if $\alpha = 0$ and $\gamma = 1$. $\square$

With respect to Lemma 3.1, an essential subgroup R has *type* (1), (2), or (3).

Definition 3.2. Let $\mathcal{F}_{\mathrm{ess}}$ be the set of essential subgroups of $G = \mathrm{G}(d, q)$, up to conjugacy, contained in D. Let $\mathcal{F}^+_{\mathrm{ess}} = \{R \in \mathcal{F} \mid R \text{ has type (3)}\}$ and $\mathcal{F}^0_{\mathrm{ess}} = \mathcal{F}_{\mathrm{ess}} \setminus \mathcal{F}^+_{\mathrm{ess}}$.

We write $d = d_0 + ew$ where w has r-adic decomposition $w = w_0 + w_1 r + \cdots + w_\ell r^\ell$. Recall that the dimension of the 1-eigenspace of the Sylow r-subgroup $D \leqslant G$ is d_0, $2d_0$, or $2d_0 + 1$, depending on the type of G, see Definition 1.4. Let $R \leqslant D$ be a radical subgroup with decomposition $R = R_0 \times \cdots \times R_s$ and $V = V_0 \perp \cdots \perp V_s$. Our previous results and [3, Lemma 11] imply that $R \in \mathcal{F}^0_{\mathrm{ess}}$ if and only if V_0 is the 1-eigenspace of D and $R = R_0 \times (C_{r^a}) \wr Q$ where Q is an essential subgroup of $S(w)$. In particular, there is a bijection between $\mathcal{F}^0_{\mathrm{ess}}$ and the conjugacy classes of essential r-subgroups of $S(w)$ contained in a fixed Sylow r-subgroup of $S(w)$. These essential subgroups of $S(w)$, up to conjugacy, and their number, are determined in [3]. Recall that $\overline{H(w)} = |\{i \in \{1, \ldots, \ell\} \mid w_i = 0\}|$ is the complementary reduced Hamming weight of w. If $\ell \geqslant 2$ with $w_{\ell-1} = 0$ and $w_\ell = 1$, then let $Y(w) = -1$, and $Y(w) = 0$ otherwise. Now [3, Theorem 5] proves the following.

Lemma 3.3. *If $\ell \geqslant 1$, then $|\mathcal{F}^0_{\mathrm{ess}}| = \ell(\ell+1)/2 + \overline{H(w)} + Y(w) - 1$, and $|\mathcal{F}^0_{\mathrm{ess}}| = 0$ otherwise.*

For an integer i let $\mathbf{e}_i = (1, \ldots, 1)$ be of weight i. If $R \in \mathcal{F}^+_{\mathrm{ess}}$, then there exist unique integers $i \geqslant 0$, $t \geqslant 0$, and $k \in \{0, 1, \ldots, r-2\}$, such that $R = R(i, t, k)$ where

$$R(i,t,k) = R_{1,0,1,\mathbf{e}_i} \times (R_{1,0,0,\mathbf{e}_{i+1}})^{r-1} \times \cdots \times (R_{1,0,0,\mathbf{e}_{i+t}})^{r-1} \times (R_{1,0,0,\mathbf{e}_{i+t+1}})^k \times R^+$$

and R^+ only contains basic subgroups $R_{1,0,0,\mathbf{e}_j}$, each of multiplicity at most $r-1$, distinct from $R_{1,0,0,\mathbf{e}_{i+n}}$ for all $n \in \{1, \ldots, t+1\}$. Observe that R^+ may contain $R_{1,0,0,\mathbf{e}_i}$. Using this decomposition, we also write $R = R^\star \times R^+$. This is implies that the Sylow r-subgroup has the form $D = (R_{1,0,0,\mathbf{e}_{i+t+1}})^{k+1} \times R^+$ (observe that $N_G(D)/D$ is an r'-subgroup) and contains no basic subgroup $R_{1,0,0,\mathbf{e}_{i+n}}$ with $n \in \{1, \ldots, t\}$.

The following lemma completes the proof of Theorem 1.5.

Lemma 3.4. $|\mathcal{F}^+_{\mathrm{ess}}| = \ell$.

Proof. If $w_i \neq 0$ with $i > 1$, then D has a subgroup $R = R(i-1, 0, w_i - 1) = R^\star \times R^+$ with $R^\star = R_{1,0,1,\mathbf{e}_{i-1}} \times (R_{1,0,0,\mathbf{e}_i})^{w_i - 1} \leqslant (R_{1,0,0,\mathbf{e}_i})^{w_i}$. Suppose there exists $n \geqslant 1$ with $i - n \geqslant 1$ such that $w_{i-n} = w_{i-n+1} = \cdots = w_{i-1} = 0$. Then also $R = R(i-1-t, t, w_i - 1) = R^\star \times R^+$ is a subgroup of D with $R^\star \leqslant (R_{1,0,0,\mathbf{e}_i})^{w_i}$ for all $t \in \{1, \ldots, n\}$. Thus for every w_k with $k \geqslant 1$ we count exactly one group of type (3), which implies the assertion of the lemma. $\square$

4. Controlled fusion

Throughout this section let $G = \mathrm{G}(d, q)$ with parameters $d, d_0, q, r, e, a, \varepsilon$ and Sylow r-subgroup D. Let V be the natural G-module. We start with two preliminary lemmas.

Lemma 4.1. *Let $V = U_0 \perp U_+$ and $\mathrm{G}(U_0) \times \mathrm{G}(U_+) \leqslant G$ where both factors have orders divisible by r, and $\dim U_+$ is even. If $e \geqslant 2$, or $e \geqslant 1$ and G is symplectic or orthogonal, then there exist $z_0 \in \mathrm{G}(U_0)$ and $z_+ \in \mathrm{G}(U_+)$ of order r, and $y \in G$ with $z_0^y = z_+$.*

Proof. Every r-element of G is semisimple and lies in a maximal torus of G. Maximal tori of classical groups are nicely described in [11], and they can, up to conjugacy, be parametrised by certain (signed) partitions of d. We use the descriptions given in [11, Section 3] and proceed as follows. We choose two maximal tori $S_0 \leqslant \mathrm{G}(U_0)$ and $S_+ \leqslant \mathrm{G}(U_+)$ which both have order divisible by r such that there exist maximal tori T_0 and T_+ of $\mathrm{G}(V)$ such that $S_0 \leqslant T_0$ and $S_+ \leqslant T_+$, the quotients T_0/S_0 and T_+/S_+ have r'-order, and T_0 and T_+ correspond to the same (signed) partition. Thus, there exists $y \in G$ with $T_0^y = T_+$, and an element of order r in S_0 is mapped under conjugation by y into S_+.

First we consider the non-orthogonal cases. Let $s = 2$ if G is symplectic, and $s = 1$ otherwise. We define $\delta = -1$ if G is unitary, $\delta = 1$ if G is linear, and $\delta = -\varepsilon$ in the symplectic case. Observe that r does not divide $q - \delta$ by our assumptions on e. We choose tori $S_0 \cong C_{q^e-\varepsilon} \times (C_{q-\delta})^{\dim U_0/s-e}$ and $S_+ \cong C_{q^e-\varepsilon} \times (C_{q-\delta})^{\dim U_+/s-e}$, which can be embedded into maximal tori T_0 and T_+ of G with

$$T_0 \cong C_{q^e-\varepsilon} \times (C_{q-\delta})^{d-e} \cong T_+.$$

Thus there is $y \in G$ with $T_0^y = T_+$, which proves the assertion for non-orthogonal groups.

For orthogonal groups we consider maximal tori of the corresponding special orthogonal subgroups. This case is more complicated because the orthogonal types (and hence the structure of maximal tori) of $\mathrm{SO}(V)$, $\mathrm{SO}(U_0)$, and $\mathrm{SO}(U_+)$ might be different and a case distinction is necessary. However, if $e \geqslant 2$, then there exist suitable u and v with $e + u + v = d$ such that we can choose maximal tori T_0 and T_+ of $\mathrm{SO}(V)$ with $T_0 \cong C_{q^e-\varepsilon} \times (C_{q+1})^u \times (C_{q-1})^v \cong T_+$. Observe that $r \nmid q^2 - 1$ since $e \geqslant 2$ is minimal with $r \mid q^{2e} - 1$, and the same argument as for the non-orthogonal case proves the assertion.

Now let $e = 1$. If G is orthogonal of type 0, then we choose maximal tori $S_0 \leqslant \mathrm{SO}(U_0)$ and $S_+ \leqslant \mathrm{SO}(U_+)$ with

$$S_0 \cong (C_{q-\varepsilon})^{u_0} \times (C_{q+\varepsilon})^{(\dim U_0 - 1)/2 - u_0} \quad \text{and} \quad S_+ \cong (C_{q-\varepsilon})^{u_+} \times (C_{q+\varepsilon})^{\dim U_+/2 - u_+}$$

where $u_0, u_+ \geqslant 1$ are as small as possible; if $\mathrm{G}(U_+)$ is of plus (minus) type, then $\dim U_+/2 - u_+$ has to be even (odd). We set $u = \max\{u_0, u_+\}$ and the same argument as before proves the assertion with $T_0 \cong (C_{q-\varepsilon})^u \times (C_{q+\varepsilon})^{d-u} \cong T_+$.

Finally, we consider $e = 1$ and G of orthogonal type $\pm$. Since U_0 and U_+ both have even dimension, the type of $\mathrm{G}(V)$ is the product of the types of $\mathrm{G}(U_0)$ and $\mathrm{G}(U_+)$. If $W \in \{U_0, U_+\}$ has dimension w, then we consider maximal tori of $\mathrm{SO}(W)$ of type $S = T \times (C_{q+\varepsilon})^{w-2}$ where T is one of $C_{q-\varepsilon} \times C_{q+\varepsilon}$ or $(C_{q-\varepsilon})^2$ such that the number of direct factors $q^i + 1$ of S with $i \geqslant 1$ is even for plus type and odd for minus type. Note that $r \nmid q^2 + 1$, and a straightforward, but technical, case distinction with respect to the possible types of $\mathrm{G}(V)$, $\mathrm{G}(U_0)$, and $\mathrm{G}(U_+)$ shows that we can apply our previous arguments with

$$T_0 \cong (C_{q-\varepsilon})^u \times (C_{q+\varepsilon})^v \times (C_{q^2+1})^w \cong T_+$$

for suitable $u \in \{1, 2\}$, v, and w. This completes the proof of the lemma. □

Lemma 4.2. *If $r^\alpha |\mathrm{GL}^\varepsilon(m, q^{er^\alpha})|_r = |\mathrm{GL}^\varepsilon(mr^\alpha, q^e)|_r$, then $\alpha = 0$.*

Proof. Let $m = m_0 + m_1 r + \cdots + m_k r^k$ be the r-adic expansion of m. The Sylow r-subgroup of $\mathrm{GL}^\varepsilon(m, q^{er^\alpha})$ is $S = S_0 \times \cdots \times S_k$, where $S_i = (R_{1,0,0,\mathbf{e}_i})^{m_i}$ (defined with respect to q^{er^α}) with

$\mathbf{e}_i = (1, \ldots, 1)$ of weight i. Analogously, the Sylow r-subgroup of $\mathrm{GL}^\varepsilon(mr^\alpha, q^e)$ is $T = T_0 \times \cdots \times T_k$, where $T_i = (R_{1,0,0,\mathbf{e}_{i+\alpha}})^{m_i}$ (defined with respect to q^e). The orders are

$$|S_i| = \left(r^{(a+\alpha)r^i+1+r+\cdots+r^{i-1}}\right)^{m_i} \quad \text{and} \quad |T_i| = \left(r^{ar^{i+\alpha}+1+r+\cdots+r^{i+\alpha-1}}\right)^{m_i}.$$

If $m_i \neq 0$, then induction on α proves $r^\alpha |S_i| \leqslant |T_i|$ with equality if and only if $\alpha = 0$. □

We now prove Theorem 1.6 and show that there exists an r-local subgroup $K \leqslant G$ which controls r-fusion if and only if D is abelian, or G is linear or unitary with $e = 1$.

Proof of Theorem 1.6. If D is abelian, then $\{D\}$ is a conjugation family for $\mathcal{F}_D(G)$, see Theorem 1.3, and $K = N_G(D)$ is r-local controlling r-fusion. If $e = 1$, then $r \mid q - \varepsilon$. Thus, if G is linear or unitary, then there exists a non-trivial central r-subgroup R; observe that $\varepsilon = -1$ in the unitary case. Clearly, $G = N_G(R)$ is r-local and controls r-fusion.

Now suppose $K \leqslant G$ with $D \leqslant K$ is r-local controlling r-fusion; we assume that $e > 1$. Note that K is contained in $N_G(O_r(K))$ and K lies in a maximal r-local subgroup $N \leqslant G$ with $N = N_G(O_r(N))$. Hence, $R = O_r(N)$ is radical. The maximality of $N_G(R)$ implies that the decomposition of R as in Remarks 2.4 or 2.5 is $R = R_0 \times R_+$ with $R_+ = (R_{m,\alpha,\gamma,\mathbf{c}})^t$ for some $m, \alpha, \gamma, \mathbf{c}, t$. Note that $N_G(R) \leqslant N_G(Z(R))$, and $Z(R) = R_0 \times (Z(R_{m,\alpha,\gamma,\mathbf{c}}))^t$ with $Z(R_{m,\alpha,\gamma,\mathbf{c}}) = R_{mr^{|\mathbf{c}|+\gamma},\alpha,0}$. Thus, by the maximality of $N_G(R)$, we can assume that $R = R_0 \times (R_{m,\alpha,0})^t$ for some m, α, t. Let $V = U_0 \perp U_+$ be the corresponding decomposition of V, thus $N_G(R) = N_0 \times N_+$, where $N_0 = \mathrm{G}(U_0)$ and $N_+ = N_{\mathrm{G}(U_+)}(R_+)$. Since $D \leqslant N_G(R)$ by assumption, we can write $D = T_0 \times T_+$, where T_0 and T_+ are Sylow r-subgroups of N_0 and N_+, respectively. Since D is a Sylow r-subgroup of G, it follows that T_0 and T_+ are also Sylow r-subgroups of $\mathrm{G}(U_0)$ and $\mathrm{G}(U_+)$, respectively. Suppose, for a contradiction, that $T_0 \neq \{1\}$. By Lemma 4.1, there exist $z_0 \in \mathrm{G}(U_0)$ and $z_+ \in \mathrm{G}(U_+)$ of order r such that $z_0^y = z_+$ for some $y \in G$. We set $P = \langle z_0, z_+\rangle$ so that $y \in N_G(P)$. Since $N_G(R)$ controls fusion, $y \in C_G(P)N_{N_G(R)}(P)$. The latter group fixes U_0 and U_+, which contradicts $z_0^y = z_+$. This contradiction implies that $T_0 = \{1\}$. Thus, $D \cong T_+$, and T_+ is a Sylow r-subgroup of both $\mathrm{G}(U_+)$ and N_+.

Note that $|N_+|_r = |N_{m,\alpha,0} \wr S(t)|_r$ with $N_{m,\alpha,0}$ as defined in Section 2.1. Since R_+ is abelian, the results of Section 2.1 show that $N_{m,\alpha,0}$ is an extension of $C_{\xi er^\alpha}$ by $\mathrm{GL}^\varepsilon(m, q^{er^\alpha})$, where $\xi = 1$ if G is linear or unitary, and $\xi = 2$ otherwise. Let $N^\star_{m,\alpha,0}$ be an extension of $C_{\xi e}$ by $\mathrm{GL}^\varepsilon(mr^\alpha, q^e)$ such that $N_{m,\alpha,0} \leqslant N^\star_{m,\alpha,0}$ and $N_+ \leqslant N^\star_{m,\alpha,0} \wr S(t) \leqslant \mathrm{G}(U_+)$. Our previous arguments show that $|N_+|_r = |\mathrm{G}(U_+)|_r$, which implies that $r^\alpha |\mathrm{GL}^\varepsilon(m, q^{er^\alpha})|_r = |\mathrm{GL}^\varepsilon(mr^\alpha, q^e)|_r$. Now Lemma 4.2 proves $\alpha = 0$, that is, we have shown that $R = R_0 \times R_+$ with $R_+ = (R_{m,0,0})^t$.

Let $P = R_0 \times (R_{1,0,0})^{mt}$ so that $R \leqslant P \leqslant D$. The group P is of radical type and $C_G(P) = \mathrm{G}(U_0) \times \mathrm{GL}^\varepsilon(1, q^e)^{mt}$ lies in $N_G(R) = \mathrm{G}(U_0) \times N_+$ since $N_+ = N_{m,0,0} \wr S(t)$ and $N_{m,0,0}$ is an extension of $C_{\xi e}$ by $\mathrm{GL}^\varepsilon(m, q^e)$. Observe that $N_G(P) = \mathrm{G}(U_0) \times M_+$ where $M_+ = N_{1,0,0} \wr S(mt)$ and $N_{1,0,0}$ is an extension of $C_{\xi e}$ by $\mathrm{GL}^\varepsilon(1, q^e)$. Since $N_G(R)$ controls r-fusion, $N_G(P) \leqslant N_G(R)$. If $t > 1$, then this immediately forces $m = 1$. Observe that $C_{\xi e} \neq \{1\}$ by our assumptions on e, and, therefore, if $t = 1$, then one can also deduce that $m = 1$. Hence, we have proved that $R = R_0 \times R_+$ with $R_+ = (R_{1,0,0})^t$.

Suppose, for a contradiction, that $t \geqslant r$ and define $P = R_0 \times (R_{1,0,0})^{t-r} \times R_{1,0,1}$ such that $P \leqslant D$ and $C_G(P) \leqslant C_G(R)$. Since $N_G(R)$ controls r-fusion, $N_G(P) \leqslant N_G(R)$. We have $N_G(P) = \mathrm{G}(U_0) \times (N_{1,0,0} \wr S(t-r)) \times N_{1,0,1}$, which implies that $P = O_r(N_G(P))$; observe that $N_{1,0,1}/R_{1,0,1}C_{1,0,1} \cong \mathrm{Sp}(2, r)$ has trivial r-core. Since $P \leqslant N_G(P) \leqslant N_G(R)$, the group P is radical in $N_G(R)$. Since $R \trianglelefteq N_G(R)$, the intersection $R \cap P$ is radical in R, see [13, Lemma 4.6]. Since R is an r-subgroup, this implies $R \cap P = R$, thus $R \leqslant P$. By construction we have $|P| < |R|$, which yields a contradiction. This proves $t < r$, and $D = R_0 \times (R_{1,0,0})^t$ is abelian. □

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